Lecture 12

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1 Examples of bases and dimensions

Last lecture we stated the result that each basis has the same number of vectors. From this result very important corollary follows.

Corollary 1.1. *If the dimension of the vector space V is equal to n: $\dim V = n$, then any n linearly independent vectors form a basis.*

Corollary 1.2. *If the dimension of the vector space V is equal to n: $\dim V = n$, then any n vectors which span all the vector space form a basis.*

Now we'll consider some examples.

Examples of bases in $\mathbb{R}^2$.

Example 1.3. *The easiest basis in $\mathbb{R}^2$ is a basis $u_1 = (1,0)$, and $u_2 = (0,1)$. These vectors are linearly independent, and span $\mathbb{R}^2$. So, we have 2 vectors in the basis of $\mathbb{R}^2$, and thus* $\dim \mathbb{R}^2 = 2$. *This particular basis is called **standard basis**.*

Now we can take any pair of linearly independent vectors and it will be a basis.

Example 1.4 (Slight modification of standard basis). *Let $u_1 = (0,5)$ and $u_2 = (1,0)$. This slight modification of the standard basis is a basis itself, since it contains 2 vectors, and is linearly independent. Moreover, we can see that any vector (a,b) can be represented as a linear combination of u_1 and u_2 in the following way:*

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{b}{5}u_1 + au_1 = \frac{b}{5}\begin{pmatrix} 0 \\ 5 \end{pmatrix} + a\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Example 1.5 (Less trivial example of basis). *Let $u_1 = (1,1)$ and $u_2 = (0,1)$. This is a basis since there are 2 vectors and they are linearly independent. We can check that these* *vectors are linearly independent by forming a linear combination which is equal to* $\mathbf{0}$ *and proving that this combination is trivial.*

$$x\begin{pmatrix}1\\1\end{pmatrix}+y\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

This is equivalent to the following system which has unique zero solution:

$$\begin{cases} x & & & = & 0 \\ x & + & y & = & 0 \end{cases}$$

So, the linear combination should be trivial, and vectors are independent.

Example 1.6 (Even more nontrivial example of basis for $\mathbb{R}^2$). *Consider the following pair of vectors $u_1 = (1, 2)$, and $u_2 = (3, 5)$. To check that this is a basis we have to prove that these 2 vectors are linearly independent. So, again we form a linear combination:*

$$x\begin{pmatrix}1\\2\end{pmatrix}+y\begin{pmatrix}3\\5\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

and try to find x's and y's —coefficients of linear combination. In order for this set to be linearly dependent, coefficients should not be zeros simultaneously. The expression above is equivalent to the following system of linear equation.

$$\begin{cases} x & + & 3y & = & 0 \\ 2x & + & 5y & = & 0 \end{cases}$$

To solve this system let's first subtract the first equation multiplied by 2 from the second one. We'll get:

$$\begin{cases} x & + & 3y & = & 0 \\ & - & y & = & 0 \end{cases}$$

Again we see that this system has unique zero-solution, and thus vectors are linearly independent. So, they form a basis for $\mathbb{R}^2$.

2 Standard bases

Let's consider different examples of vector spaces and find a basis of it.

2.1 $\mathbb{R}^n$

The basis of $\mathbb{R}^n$ — the set of all n-tuples is the following. It contains the following n vectors:

$$e_1 = (1, 0, 0, \dots, 0, 0), \quad e_2 = (0, 1, 0, \dots, 0, 0),$$
$$e_3 = (0, 0, 1, \dots, 0, 0), \quad \dots,$$
$$e_{n-1} = (0, 0, 0, \dots, 1, 0), \quad e_n = (0, 0, 0, \dots, 0, 1)$$

where the vector e_i has 1 on the i-th place and zeros on all other places. One can easily check that these vectors are linearly independent. So,

$$\dim \mathbb{R}^n = n$$

2.2 $\mathbb{P}_n$

The basis of $\mathbb{P}_n$ — the set of all polynomials of degree less or equal to n is the following. It contains the following $n+1$ vectors:

$$e_0 = 1, \quad e_1 = t, \quad , e_2 = t^2, \quad \ldots,$$
$$e_{n-1} = t^{n-1}, \quad t_n = t^n$$

where the vector e_i is the i-th power of t. One can easily check that these vectors are linearly independent. So,

$$\dim \mathbb{P}_n = n+1$$

2.3 $M_{m,n}$

The basis of $M_{m,n}$ — the set of all $m \times n$-matrices is the following. It contains the following mn vectors:

$$e_{1,1} = I_{1,1}, \quad e_{1,2} = I_{1,2}, \quad , \ldots,$$
$$e_{i,j} = I_{i,j}, \quad \ldots$$

where the vector e_i is the matrix with 1 on the (i,j)-th place and 0's on all other places. One can easily check that these vectors are linearly independent. So,

$$\dim M_{m,n} = mn$$

For example, the basis of $M_{2,2}$ consists of the following $2 \times 2 = 4$ vectors:

$$e_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{1,2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{2,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

We see that each 2×2-matrix can be represented as a linear combination of these 4 matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ae_{1,1} + be_{1,2} + ce_{2,1} + de_{2,2} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

All bases given in this section are called **standard bases**.

3 Dimension and basis of the span

We'll start this section with an example to show what are we going to find and which problem do we want to solve.

Example 3.1. *Consider the vector space* $\mathbb{R}^2$*, and let* $u_1 = (1, 0)$*,* $u_2 = (0, 1)$*, and* $u_3 = (1, 1)$*. What is the span of these 3 vectors? It is obvious that these vectors spans all* $\mathbb{R}^2$ *since each vector from* $\mathbb{R}^2$ *can be represented as a linear combination of them. But* $\dim \mathbb{R}^2 = 2$*. So, we see that* $\dim \operatorname{span}(u_1, u_2, u_3) = 2$ *— not equal to the number of vectors.*

Example 3.2. *Consider the vector space* $\mathbb{R}^2$*, and let*

$$\begin{aligned} u_1 &= (1, 1, 3, 2), \\ u_2 &= (0, 2, -1, 1), \\ u_3 &= (1, 3, 2, 3), \\ u_4 &= (1, -1, 4, 1) \end{aligned}$$

What is the span of these 4 vectors? We can easily see that

$$\begin{aligned} u_3 &= u_1 + u_2, \qquad \textit{and} \\ u_4 &= u_1 - u_2. \end{aligned}$$

So, $u_3 \in \operatorname{span}(u_1, u_2)$ *and* $u_4 \in \operatorname{span}(u_1, u_2)$*. So we conclude that we don't need these 2 vectors* u_3 *and* u_4 *— everything which can be expressed as a linear combination of* u_1*,* u_2*,* u_3*, and* u_4 *can be expressed as a linear combination of* u_1 *and* u_2 *only:*

$$\begin{aligned} v &= au_1 + bu_2 + cu_3 + du_4 \\ &= au_1 + bu_2 + c(u_1 + u_2) + d(u_1 - u_2). \end{aligned}$$

And vectors u_1 *and* u_2 *are linearly independent. So, in this example* $\operatorname{span}(u_1, u_2, u_3, u_4) = \operatorname{span}(u_1, u_2)$*, and* $\dim \operatorname{span}(u_1, u_2, u_3, u_4) = 2$*.*

So, our problem is the following. Suppose, we're given set of n vectors in V. We want to find maximal linearly independent subset — it is a basis for the span of all these n vectors, and the number of vectors in this subset is the dimension of the span.

We will formulate the problem in a strict way now.

Let V be a vector space. Let u_1, u_2, $\dots$, u_m — be vectors in V. Consider the span of these vectors $\langle u_1, u_2, \dots, u_m \rangle$ — all vectors which can be represented as a linear combination of vectors u_i's. Our problem is

- to find the dimension of a span
- to find the basis of span

Now, when we stated our problem we can develop the theory which will help us to solve it.

3.1 Elementary operations with vectors

Let we have m vectors from the vector space V: $u_1, u_2, \dots, u_m \in V$. Consider the span of them and let's see does it change when we'are changing the vectors. These changes of vectors will be called **elementary operations** and there will be 3 types of them.

Type 1. Interchanging of vectors. Let's interchange vectors u_i and u_j, i.e. let's change the order of them. Then the span will not change:

$$\langle u_1, \dots, u_i, \dots, u_j, \dots, u_m \rangle = \langle u_1, \dots, u_j, \dots, u_i, \dots, u_m \rangle$$

It means that everything which can be expressed as a linear combination of the vectors in unchanged order, can be expressed as a linear combination of vectors in different order — it is obvious!

So, this elementary operation being applied to the set of vectors doesn't change its span.

Type 2. Multiplication of a vector by a number. Let $c \neq 0$, and let's multiply the vector u_i by c. Then

$$\langle u_1, \dots, u_i, \dots, u_m \rangle = \langle u_1, \dots, cu_i, \dots, u_m \rangle$$

To prove it let's consider a vector which can be represented as linear combination of unchanged system of vectors:

$$v = x_1 u_1 + \dots + x_i u_i + \dots + x_m u_m \text{ — linear combination of unchanged vectors.}$$

Then

$$v = x_1 u_1 + \dots + \frac{x_i}{c}(cu_i) + \dots + x_m u_m \text{ — linear combination of changed vectors.}$$

So, if we multiply the vector by a number, we should divide the coefficient by the same number. Moreover, if a vector can be represented as a linear combination of the changed vectors, it can be represented as a linear combination of unchanged vectors: if

$$v = y_1 u_1 + \dots + y_i(cu_i) + \dots + y_m u_m \text{ — linear combination of changed vectors}$$

then

$$v = y_1 u_1 + \dots + (cy_i)u_i + \dots + x_m u_m \text{ — linear combination of unchanged vectors.}$$

So, this elementary operation being applied to the set of vectors doesn't change its span.

Type 3. Addition of a vector multiplied by a number to some other vector. Let's add vector u_j multiplied by some number k to the vector u_i. Now instead of the old set of vectors $\{u_1, \dots, u_i, \dots, u_j, \dots, u_m\}$ we'll have a set $\{u_1, \dots, u_i + ku_j, \dots, u_j, \dots, u_m\}$. Then the spans of these to sets of vectors are the same:

$$\langle u_1, \dots, u_i, \dots, u_j, \dots, u_m \rangle = \langle u_1, \dots, u_i + ku_j, \dots, u_j, \dots, u_m \rangle$$

Let's prove it. Let a vector v can be expressed as linear combination of the vectors from unchanged set:

$$v = x_1u_1 + \cdots + x_iu_i + \cdots + x_ju_j + \cdots + x_mu_m.$$

Then v can be represented as a linear combination of the vectors from the changed set in the following way:

$$v = x_1u_1 + \cdots + x_i(u_i + ku_j) + \cdots + (x_j - kx_i)u_j + \cdots + x_mu_m.$$

We'll give an example. Vector $(2, 4)$ can be represented in the following way:

$$(2, 4) = 2(1, 0) + 4(0, 1).$$

Now let's add the second vector multiplied by 3 to the first vector — we'll have vectors $(1, 3),\ (0, 1)$. Let's represent $(2, 4)$ as a linear combination of these changed vectors.

$$(2, 4) = 2(1, 3) + (4 - 3 \cdot 2)(0, 1) = 2(1, 3) - 2(0, 1).$$

In the similar way we can prove that any vector which can be represented as a linear combination of changed set of vectors, can be represented as a linear combination of the initial set of vectors.

So, this elementary operation being applied to the set of vectors doesn't change its span.